Time recurrent behaviour in the nonlinear Schrodinger equation

1980 J. Phys. A: Math. Gen. 13 2395

(http://iopscience.iop.org/0305-4470/13/7/022)

J. Phys. A: Math. Gen. **13** (1980) 2395–2399. Printed in Great Britain

Time recurrent behaviour in the nonlinear Schrödinger equation

G Rowlands
Department of Physics, University of Warwick, Coventry, CV4 7AL, England

Received 29 October 1979, in final form 28 January 1980

Abstract. An analytic treatment of the time behaviour of spatially periodic solutions of the nonlinear Schrödinger equation is given. This predicts periodic time evolution in qualitative agreement with recent numerical results.

1. Introduction

In a recent paper, Yuen and Ferguson (1978) studied, numerically, the time evolution of solutions of the nonlinear Schrödinger equation subject to spatial periodic boundary conditions. Their results showed very interesting time recurrence phenomena. In particular, for the case which they label as simple, the time evolution is periodic. It is the purpose of this paper to show how this periodicity arises analytically and to obtain an estimate for the period.

2. Basic method

In the notation of Yuen and Ferguson the nonlinear Schrödinger equation takes the form

$$\mathrm{i}\,\partial A/\partial t-\tfrac{1}{8}\,\partial^2 A/\partial x^2-\tfrac{1}{2}|A|^2A=0.$$

It is convenient to write $A=\psi(x,t)\exp[-(\mathrm{i}/2)a_0^2t]$, in which case

$$\mathrm{i}\,\partial\psi/\partial t+\tfrac{1}{2}a_0^2\psi-\tfrac{1}{8}\,\partial^2\psi/\partial x^2-\tfrac{1}{2}|\psi|^2\psi=0. \tag{2.1}$$

This equation is to be solved subject to periodic boundary conditions so we write

$$\psi(x,t)=\sum_{n=0}^{\infty}\phi_n(t)\cos(nk_0x), \tag{2.2}$$

and restrict attention to symmetric solutions in accord with the numerical simulations.

A linearised analysis of (2.1) shows that $\phi_0=a_0$ and $\phi_n=\alpha\exp(\lambda_nt)$ where

$$\lambda_n^2=\tfrac{1}{8}n^2k_0^2(a_0^2-n^2k_0^2/8). \tag{2.3}$$

The 'simple case', as considered by Yuen and Ferguson, corresponds to the situation where $\lambda_1^2>0$ but all other λ_n are purely imaginary, i.e. there is only one unstable mode.

In such a case one could imagine that in the full nonlinear case the $n=1$ mode would act as a driver to the other modes such that these modes follow the time evolution of ϕ_1.

Yuen and Ferguson in fact point out that their numerical results show that the $n \neq 1$ modes appear phase locked to the $n = 1$ mode. The analysis given in this paper is based on this observation. This concept of phase locking or 'enslaving' adiabatically has been discussed in detail in other branches of science (see for example Haken (1977), particularly ch 7).

Mathematically one proceeds as follows. If $\psi(x, t)$, as given by (2.2), is substituted into (2.1) one obtains an infinite set of coupled equations of the form

$$\mathrm{d}\phi_n/\mathrm{d}t + \lambda_n\phi_n = F_n(\phi_m), \tag{2.4}$$

where F is a nonlinear function of the ϕ_n. Formally these may be solved to give

$$\phi_n = \exp(-\lambda_n t)\int^t F_n(\phi_m)\exp(\lambda_n t')\,\mathrm{d}t'.$$

One now makes the basic assumption that the time evolution of the problem proceeds on a timescale characterised by $1/\lambda_1$. If we demand that $\lambda_1 \ll |\lambda_n|$, then $F_n(\phi_m)$ varies slowly with time compared to $\exp(\lambda_n t)$ and may be taken outside the integral to give, for $n > 1$,

$$\phi_n \approx F_n(\phi_m)/\lambda_n. \tag{2.5}$$

This is equivalent to neglecting the time derivative of ϕ_n in equation (2.4). For the $n = 0$ mode the time derivative must be kept since $\lambda_0 \equiv 0$. In principle equations (2.5) may be solved and all the ϕ_n, $n > 1$, expressed in terms of ϕ_0 and ϕ_1. These values are then substituted into equation (2.4) for $n = 0$, $n = 1$ to give two coupled ordinary nonlinear differential equations for ϕ_0 and ϕ_1.

An alternative method is to proceed as follows. The approximation which led to (2.5) is seen to be equivalent to solving the equation

$$\frac{1}{8}\frac{\partial^2\psi}{\partial x^2} - \tfrac{1}{2}a_0^2\psi + \tfrac{1}{2}|\psi|^2\psi = \mathrm{i}\frac{\mathrm{d}\phi_0}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}\phi_1}{\mathrm{d}t}\cos(k_0x)$$

subject to the condition that ψ is periodic with period $2\pi/k_0$. This is equivalent to solving the ordinary differential equation

$$\tfrac{1}{8}\mathrm{d}^2\psi_\mathrm{e}/\mathrm{d}x^2 - \tfrac{1}{2}a_0^2\psi_\mathrm{e} + \tfrac{1}{2}|\psi_\mathrm{e}|^2\psi_\mathrm{e} = \eta_1 + \eta_2\cos(k_0x), \tag{2.6}$$

where η_1 and η_2 are to be treated as constants, and then imposing the conditions

$$\mathrm{i}\frac{\mathrm{d}\phi_0}{\mathrm{d}t} \equiv \eta_1 = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{k_0}{2\pi}\int_0^{2\pi/k_0}\psi_\mathrm{e}(x, \eta_1, \eta_2)\,\mathrm{d}x\right) \tag{2.7}$$

and

$$\mathrm{i}\frac{\mathrm{d}\phi_1}{\mathrm{d}t} \equiv \eta_2 = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{k_0}{\pi}\int_0^{2\pi/k_0}\psi_\mathrm{e}(x, \eta_1, \eta_2)\cos(k_0x)\,\mathrm{d}x\right). \tag{2.8}$$

Equations (2.7) and (2.8) constitute two coupled ordinary nonlinear differential equations for η_1 and η_2, and describe the time evolution of the system. The time evolution of the amplitude of the higher modes, $\phi_n(t)$, is readily obtained by decomposing ψ_e into its Fourier modes, the time dependence being through that of ϕ_0 and ϕ_1 only.

Unfortunately it has not been possible to obtain an exact solution to (2.6), and hence the forms for the equations for η_1 and η_2 remain unknown. However in the limit as $\lambda_1 \to 0$ it is possible to solve (2.6) by expansion and subsequently obtain approximate equations for η_1 and η_2. This is carried out in the next section.

3. Approximate time-dependent equations

For λ_1 small, the growth rate is small and one expects the nonlinear effects also to be small. Thus we consider the solution of (2.6) in this weakly nonlinear case by formally expanding ψ_e as a power series,

$$\psi_e = a_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots,$$

and treating $\eta_2 = O(\epsilon)$ and η_1 of higher order. In this way one generates a hierarchy of equations of the general form

$$\tfrac{1}{8}\,d^2\psi_n/dx^2 + \tfrac{1}{2}a_0(\psi_n + \psi_n^x) = S_n \tag{3.1}$$

where S_n depends on ψ_m with $m < n$ and possibly on η_1 and η_2. In particular $S_1 = \eta_2\cos(k_0x)$, so writing $\psi_n = A\cos(k_0x)$ one finds

$$A = (2/D)(\alpha\eta_2 - \beta\eta_2^x) \tag{3.2}$$

where $\alpha = a_0^2 - k_0^2/4$, $\beta = a_0^2$ and $D = \alpha^2 - \beta^2$.

If we now treat η_1 as being of higher order than ϵ^2 then

$$S_2 = -\tfrac{1}{4}a_0A(A + 2A^x)(1 + \cos 2k_0x).$$

From (3.1) one obtains

$$\tfrac{1}{8}\,d^2(\psi_2 - \psi_2^x)/dx^2 = -\tfrac{1}{4}a_0(1 + \cos 2k_0x)[A^2(A^x)^2],$$

which when integrated gives rise to a non-periodic variation with x. Such a solution is not allowed and can be avoided by considering η_1 to be of order ϵ^2 and using this freedom to remove this secular behaviour. Thus we choose

$$\eta_1 - \eta_1^x = \tfrac{1}{4}a_0[A^2 - (A^x)^2].$$

It is then found that

$$\psi_2 = B + F\cos(2k_0x)$$

where

$$B = \frac{(\eta + \eta^x)}{2a_0^2} - \frac{[A^2 + (A^x)^2 + 4AA^x]}{8a_0}$$

and

$$F = \frac{a_0}{4}\left(\frac{[A^2 - (A^x)^2]}{k_0^2} - \frac{[A^2 + (A^x)^2 + 4AA^x]}{2a_0^2 - k_0^2}\right).$$

To next order

$$S_3 = -a_0(\psi_1\psi_2 + \psi_2\psi_1^x + \psi_1\psi_2^x) - \tfrac{1}{2}\psi_1^2\psi_1^x,$$

and this expression contains terms proportional to $\cos(k_0x)$ and $\cos(3k_0x)$. We are only interested in the terms proportional to $\cos(k_0x)$ and we find, writing $\psi_3 = H\cos(k_0x) +$ terms proportional to $\cos(3k_0x)$, that

$$H = (2/D)(\alpha T_3 - \beta T_3^x)$$

where

$$T_3 = -[a_0B(2A + A^x) + \tfrac{1}{2}a_0(AF + A^xF + AF^x) + \tfrac{3}{8}A^2A^x].$$

Thus to this order we may write $\psi_e = (a_0 + B) + (A + H)\cos(k_0 x) + \text{harmonics}$, in which case (2.7) reduces to

$$\eta_1 = \mathrm{i}\, \mathrm{d}B/\mathrm{d}t \tag{3.3}$$

whilst (2.8) reduces to

$$\eta_2 = \mathrm{i}\, \mathrm{d}(A + H)/\mathrm{d}t. \tag{3.4}$$

We notice from the definition of B that it is real and hence, from above, $\eta_1 = -\eta_1^x$ and

$$\eta_1 = \tfrac{1}{8} a_0 [A^2 - (A^x)^2].$$

Thus we see that the time dependence of η_1, and hence ϕ_0, is given algebraically in terms of ϕ_1, and thus even ϕ_0 is phase locked to the time dependence of ϕ_1. Equation (3.3) is now redundant and is in fact equivalent to the linearised version of (3.4).

Equations (3.2) and (3.4) now constitute a system of equations for the time evaluation of η_2 or A. It is convenient to express these equations in terms of $X = (A + A^x)/2$ and $Y = -\mathrm{i}(A - A^x)/2$, in which case we find that

$$(\alpha + \beta) X = -2 \frac{\mathrm{d}}{\mathrm{d}t} \{ Y[1 + g(X^2, Y^2)] \}, \tag{3.5}$$

$$(\alpha - \beta) Y = 2 \frac{\mathrm{d}}{\mathrm{d}t} \{ X[1 + h(X^2, Y^2)] \}, \tag{3.6}$$

where

$$g = \frac{1}{(\alpha - \beta)} \left(\frac{(3X^2 - Y^2)}{4} + \frac{a_0^2 [k_0^2 (5X^2 + Y^2) - 4 a_0^2 X^2]}{2 k_0^2 (2 a_0^2 - k_0^2)} \right)$$

and

$$h = \frac{1}{(\alpha + \beta)} \left(\frac{3(5X^2 + Y^2)}{4} + \frac{a_0^2 [k_0^2 (9X^2 + 5Y^2) - 4 a_0^2 Y^2]}{2 k_0^2 (2 a_0^2 - k_0^2)} \right).$$

Since we are primarily interested in situations where λ_1 is small, we may put $k_0^2 = a_0^2(\lambda_1 = 0)$ in the above expressions for g and h except in the prefactor $1/(\alpha + \beta)$. This gives

$$g = \frac{1}{(\alpha - \beta)} \left(\frac{9X^2 - 8Y^2}{24} \right)$$

and

$$h = \frac{1}{(\alpha + \beta)} \left(\frac{3(8X^2 + Y)^2}{8} \right).$$

Equations (3.5) and (3.6) may be put into a more transparent form by introducing $\bar{X} = X[1 + h(X, Y)]$ which, since we have considered h to be small, may be expressed as $X = \bar{X}[1 - h(\bar{X}, \bar{Y})]$. Using this transformation, and replacing $\bar{X}$ by $\bar{X} a_0$ and finally dropping the bars gives the equations

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = -X\delta \left(1 - \frac{3(8X^2 + Y^2)}{16\delta} \right), \tag{3.7}$$

and

$$\frac{dX}{d\tau} = -Y\left(1 + \frac{(9X^2 - 8Y^2)}{48}\right), \tag{3.8}$$

where $\tau = a_0^2 t$ and $k_0^2 = 8a_0^2(1-\delta)$, so that $\frac{1}{2} > \delta > 0$.

The critical points of the above equations are $X = Y = 0$ and $X^2 = 2\delta/3$, $Y = 0$. The first corresponds to the usual equilibrium point and is unstable with a growth rate equal to $1/a_0^2\sqrt{\delta}$, whilst about the other critical point the solution is oscillating with period of order

$$T = (2\pi)^{1/2}/a_0^2\delta^{3/2}. \tag{3.9}$$

A phase plane analysis of equations (3.7) and (3.8) shows that all solutions correspond to closed orbits and hence to periodic solutions. In general the period will depend on the initial conditions but for those near the critical point the period is given by (3.9).

Thus we have shown that the assumption of phase-locking of modes to the fundamental can explain the time-recurrent behaviour of the spatially periodic solutions of the nonlinear Schrödinger equation, at least in the 'simple' case.

4. Complex recurrences

Besides the 'simple' case discussed above, Yuen and Ferguson give results for situations which they call complex. These correspond to values of a_0^2 where more than one value of λ_n is positive, that is more than one mode is linearly unstable. In the light of the work presented in this paper one expects that one must treat all the unstable modes on an equal footing, but allow the stable ones to phase lock to them. Mathematically this assumption is expressed by including on the right-hand side of (2.6) a contribution from all the unstable modes, and introducing equations analogous to (2.7) and (2.8) for each of these modes. Then, analogous to (3.7) and (3.8), one expects two new coupled equations for each new unstable mode with the possibility of coupling between all equations. In general one would no longer expect exact periodicity but some form of recurrent behaviour is still to be expected. The exact nature will of course depend on the form of the coupling between the equations.

5. Discussion

An analytic treatment of the time evolution of spatially periodic solutions of the nonlinear Schrödinger equation has been given. It is based on the assumption that the linearly stable modes are phase locked to the unstable ones. Solutions varying periodically with time are obtained in qualitative agreement with numerical simulations.

References

Haken H 1977 *Synergetics* (Berlin: Springer-Verlag)
Yuen H C and Ferguson W E Jr 1978 *Phys. Fluids* **21** 1275